In the Name of GOD

Vector space / Signal Space

In this session we will talk about vector & signal space. As you may know, we can show a set of det. band-pass signals in a signal space and corresponding vector space, call it signal constellation (in digital modulated signal) So, it is necessary to have an explanation about vector space & signal space and how these are correspond to each other for a set of det. signals.

Vector space & its characterstics & some definitions

In an n-dimentional vector space we work with vectors such as $\underline{v}$, in which

$$\underline{v} = [v_1, v_2, \ldots, v_n]^T$$

($\underline{v}$ is a column vector)

It is an standard notation in Comm System to show the vectors as Column vectors

$$v_i \in \mathbb{C}$$

(set of Real & Complex numbers)

As you may know, in an n_dim Vector space, we can write any vector $\underline{v}$, as the linear combination of the vector space's basis vectors, means, $\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n$

$$\Longrightarrow \quad \underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + \dots + v_n \underline{e}_n$$

(In any n-dim vector space, we have n linear Independent bases vectors, so that all vectors in the vector space can be represent by the linear Combination of these basis vector)

$\{\underline{e}_i\}_{i=1}^{n}$ are the basis vectors in our vector space.

Also $\{\underline{e}_i\}_{i=1}^{n}$ are orthonormal basis vectors.

In this part we want to explain about some operations and definitions and some relations in vector spaces

1) Inner Product of two vectors $\underline{v}_1$ & $\underline{v}_2$

$$\langle \underline{v}_1 , \underline{v}_2 \rangle \triangleq \sum_{i=1}^{n} v_{1_i} v_{2_i}^{*} = \underline{v}_1 \cdot \underline{v}_2^{*} = \underline{v}_2^{H} \underline{v}_1 = \langle \underline{v}_2 , \underline{v}_1 \rangle^{*}$$

$$\underline{v}_1 = (v_{1_1} , v_{1_2} , \dots , v_{1_n})^{T}$$

$$\underline{v}_2 = (v_{2_1} , v_{2_2} , \dots , v_{2_n})^{T}$$

If $\underline{v}_1$ & $\underline{v}_2$ are real vectors then the inner product of two vectors has symmetry charaterstic, $\langle \underline{v}_1 , \underline{v}_2 \rangle = \langle \underline{v}_2 , \underline{v}_1 \rangle$

2) Vector Norm

Using the inner product definition, we can define the vector norm for any vectors $\underline{v}_1$, as

$$\underbrace{\| \underline{v}_1 \|}_{\text{Vector Norm}} = \langle \underline{v}_1 , \underline{v}_1 \rangle^{1/2} = \left[\sum_{i=1}^{n} | v_{1_i} |^2 \right]^{1/2}$$

3) orthogonality of two vectors $\underline{v}_1$ & $\underline{v}_2$

Based the definition of inner product, we can define the concept of orthogonality for any two vectors, $\underline{v}_1$ & $\underline{v}_2$

$$\text{if} \quad \langle \underline{v}_1 , \underline{v}_2 \rangle = 0 \quad \Longleftrightarrow \quad \underline{v}_1 \perp \underline{v}_2$$

$\underline{v}_1$ and $\underline{v}_2$ are orthogonal

4) An orthogonal set of vectors

A set of vectors $\underline{v}_1 , \underline{v}_2 , \ldots , \underline{v}_k$ is called an orthogonal set of vectors, if any two vectors of this set are orthogonal

$\forall\, i \,\&\, j \,,\; i \neq j \;;\quad \underline{v}_i \perp \underline{v}_j \quad \text{or} \quad \langle \underline{v}_i , \underline{v}_j \rangle = 0$

$i, j \in \{1, 2, .., k\}$

$\Longrightarrow \quad \{\underline{v}_i\}_{i=1}^{k}$ is an orthogonal set of vectors.

5) orthonormal set of vectors

A set of vectors $\{\underline{v}_i\}_{i=1}^{k}$ is called an orthonormal set if any two vectors of this set are orthogonal and with the norm equal to 1 (unit norm)

1) $\forall\ i\ \&\ j\ ;\ i \neq j \quad ; \quad \underline{v}_i \perp \underline{v}_j$

$i, j \in \{1, .., k\}$

$\iff \{\underline{v}_i\}_{i=1}^{k}$ is ortho-normal

2) $\forall\ i \in \{1, .., k\} \quad ; \quad \| \underline{v}_i \| = 1$

or

if $\langle \underline{v}_i , \underline{v}_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \iff \{\underline{v}_i\}_{i=1}^{k}$ is orthonormal

$i\ \&\ j \in \{1, .., k\}$

6) linear Independency of a set of Vectors

As you may know, we call the set of vector $\{\underline{v}_i\}_{i=1}^{k}$ a linearly Independent vectors if we can not represent any of these vectors as the linear Combination of the other vectors. So,

If the equation of $a_1\underline{v}_1 + a_2\underline{v}_2 + \cdots + a_k\underline{v}_k = \underline{0}$ just has the unique answer of zero, mean $a_1 = a_2 = \cdots = a_k = 0$ then vectors $\{\underline{v}_i\}_{i=1}^{k}$ are linearly Independent.

otherwise, vectors $\{\underline{v}_i\}_{i=1}^{n}$ are linearly dependent.

In the next part we want to introduce some inequalities in the vector space.

7) Tri-angular inequality

In any vector space, we have the Tri-angular Inequality as below, for any vectors $\underline{v}_1$ & $\underline{v}_2$

$$\|\underline{v}_1 + \underline{v}_2\| \leq \|\underline{v}_1\| + \|\underline{v}_2\|$$

in which eauality happens, when $\underline{v}_1$ and $\underline{v}_2$ are in the same dirctions, means

$$\exists\, a \in \mathbb{R}^+ \quad ; \quad \underline{v}_1 = a\,\underline{v}_2$$

(Prove this inquality an an assighment)

$$\langle \underline{v}_1, \underline{v}_2 \rangle = \|\underline{v}_1\| \, \|\underline{v}_2\| \cos \angle \underline{v}_1, \underline{v}_2$$

$$\langle \underline{v}_1, \underline{v}_2 \rangle = \langle \underline{v}_2, \underline{v}_1 \rangle^* \implies Re\left\{\langle \underline{v}_1, \underline{v}_2 \rangle\right\} = \frac{1}{2}\left[\langle \underline{v}_1, \underline{v}_2 \rangle + \langle \underline{v}_2, \underline{v}_1 \rangle\right]$$

8) Cauch–Tshowartz Inequality (Tshowartz Inequality)

This inequality is one of the useful and helpful inequality in our work.

$$\left| \langle \underline{v}_1 , \underline{v}_2 \rangle \right| \leq \| \underline{v}_1 \| \, \| \underline{v}_2 \|$$

(Prove this inequality as an assignment)

we have equality when $\underline{v}_1$ & $\underline{v}_2$ are in the same line, means

$$\exists \, a \in \mathbb{C} \; ; \quad \underline{v}_1 = a \underline{v}_2$$

In vector space we have also pysagorean eaquality for any orthogonal vectors $\underline{v}_1$ and $\underline{v}_2$ as below,

$$\| \underline{v}_1 + \underline{v}_2 \|^2 = \| \underline{v}_1 \|^2 + \| \underline{v}_2 \|^2$$

In the next part we want to explain about Gram-Schemit Algorithm. This algorithm help us to find a basis vector set for the vector space of any set of vectors such as

$$\{ \underline{v}_1, \dots, \underline{v}_m \} = \{ \underline{v}_i \}_{i=1}^{m}$$

As you may know any set of vectors $\{\underline{v}_i\}_{i=1}^{m}$, make a (sub) vector space in the n-dim. vector space. If we find the basis vector set for this (sub) vector space, then we can represent all the vectors in this (sub) vector space, by the linear combination of corresponding basis vectors. And so, we can recognize this (sub) vector space.

So, Gram-Schmidt algorithm is one of important tools that help us analysing our (sub) vector space (and also corresponding signal space)

9) Gram-Schemit Algorithm

Suppose a set of m vectors as $\{\underline{v}_i\}_{i=1}^{m}$.

we want to find a set of basis vectors for the
orthonormal

(sub)vector space made by $\{\underline{v}_i\}_{i=1}^{m}$. Gram-Schemit algorithm, help us to reach this aim as follows

step 1) Select a vector from the set $\{\underline{v}_i\}_{i=1}^{m}$, normalize it and take it as the first basis vector.

$$\underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}$$ (So $\underline{u}_1$ will be a unit norm vector)

the first basis vector

step 2) Select a vector from the set $\{\underline{v}_i\}_{i=1}^{m}$ thas has not selected in the previous steps. Reduce the projection of this vector onto previous basis vectors from it. Normalize the result vector and take it as the next basis vector.

for example in step 2, we select $\underline{v}_2$

$$\underline{v}_2' = \underline{v}_2 - \underbrace{\langle \underline{v}_2 , \underline{u}_1 \rangle \underline{u}_1}_{\text{the projection of } \underline{v}_2 \text{ onto } \underline{u}_1} \; :$$

$$\underline{u}_2 = \frac{\underline{v}_2'}{\| \underline{v}_2' \|}$$

or in the next step, we select $\underline{v}_3$

$$\underline{v}_3' = \underline{v}_3 - \underbrace{\langle \underline{v}_3 , \underline{u}_1 \rangle \underline{u}_1}_{\text{projection of } \underline{v}_3 \text{ onto } \underline{u}_1} - \underbrace{\langle \underline{v}_3 , \underline{u}_2 \rangle \underline{u}_2}_{\text{Projection of } \underline{v}_3 \text{ onto } \underline{u}_2}$$

$$\underline{u}_3 = \frac{\underline{v}_3'}{\| \underline{v}_3' \|}$$

step 3) Repeat step 2 till all the vectors in set $\{\underline{v}_i\}_{i=1}^{m}$ take part in the algorithm.

for example in the k^{th} repetition we have

$$\underline{v}_k' = \underline{v}_k - \langle \underline{v}_k, \underline{u}_1 \rangle u_1 - \langle \underline{v}_k, \underline{u}_2 \rangle \underline{u}_2 - \cdots - \langle \underline{v}_k, \underline{u}_{k-1} \rangle u_{k-1}$$

$$\underline{v}_k' = \underline{v}_k - \sum_{i=1}^{k-1} \langle \underline{v}_k, \underline{u}_i \rangle u_i$$

$$\underline{u}_k = \frac{\underline{v}_k'}{\|v_k'\|} \longrightarrow k^{th} \text{ basis function}$$

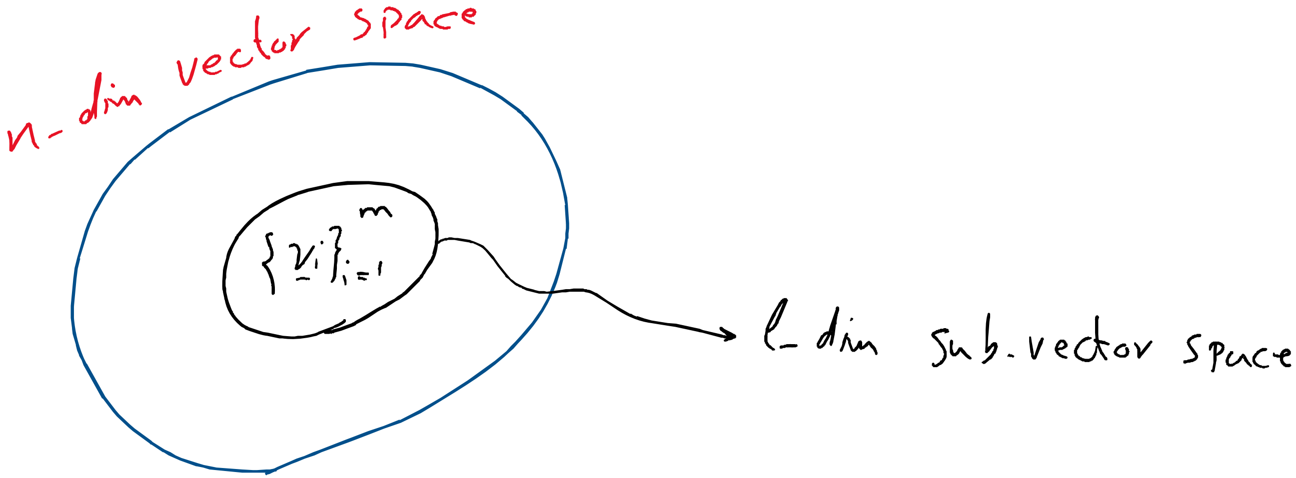
At the end of algorithm we have a set of orthonormal basis vectors as $\{\underline{u}_i\}_{i=1}^{l}$ for the (sub)vector space made by the set of $\{\underline{v}_i\}_{i=1}^{m}$.

such that $l \leq m$

Maybe in the signal set of $\{\underline{v}_i\}_{i=1}^{m}$, we have a vector $\underline{v}_r$ such that it is equal a linear combination of the other vectors in the set of $\{\underline{v}_i\}_{i=1}^{m}$. So, when we

Select $\underline{v}_r$ in Gram-Schemit algorithm, we have $\underline{v}'_r = \underline{0}$ means tha $\underline{v}_r$ is equal to linear Combination of the other vectors in set $\{\underline{v}_i\}_{i=1}^{m}$ and does not give us any basis vector. So, we can ommit $\underline{v}_r$ from the algorithm and repeat the algorithm for the remaining vectors of set $\{\underline{v}_i\}_{i=1}^{m}$. That is why ℓ (the number of orthonormal basis vectors) is always smaller or equal to m (the number of initial vector set made the subvector space) we have $\ell \leq m$

On the other hand, we know that sub-vector space made by vector set $\{\underline{v}_i\}_{i=1}^{m}$ is a sub-vector space of an n-dimentional vector space. So, we have $\ell \leq n$



So, we can conclude that $\ell \leq \min(m, n)$

If the vectors $\{\underline{v}_i\}_{i=1}^{m}$ are linear independent vectors then $\ell = m$, because no vector in the of $\{\underline{v}_i\}_{i=1}^{m}$ is not a linear combination of the other vectors in the set. So, in any repitition of Gram-Schemit algorithm we gain a basis vector. and we have m

orthonormal basis vectors.

In the next session we will talk about signal space & the relation between signal and vector space.

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